Chapter 8: Estimating with Confidence

Section 8.3

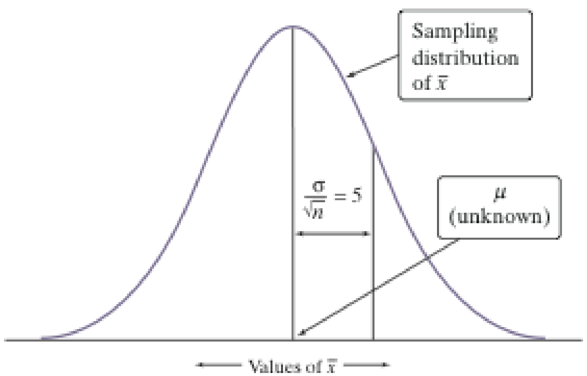
Estimating a Population Mean

The One-Sample *z* Interval for a Population Mean

In Section 8.1, we estimated the “mystery mean” μ (see page 468) by constructing a confidence interval using the sample mean = 240.79.

To calculate a 95% confidence interval for μ , we use the familiar formula:

estimate ± (critical value) • (standard deviation of statistic)



$$\bar{x} \pm z^* \cdot \frac{\sigma}{\sqrt{n}} = 240.79 \pm 1.96 \cdot \frac{20}{\sqrt{16}}$$

$$= 240.79 \pm 9.8$$

$$= (230.99, 250.59)$$

One-Sample *z* Interval for a Population Mean

Choose an SRS of size *n* from a population having unknown mean μ and known standard deviation σ. As long as the Normal and Independent conditions are met, a level *C* confidence interval for μ is

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$$

The critical value *z** is found from the standard Normal distribution.

Example – One Sample Z-Interval for a Population Mean

A bottling machine is operating with a standard deviation of 0.12 ounce. Suppose that in an SRS of 36 bottles the machine inserted an average of 16.1 ounces into each bottle.

A) Estimate the mean number of ounces in all the bottles this machine fills

$$Since\ \mu = \bar{x}, then$$

$$\bar{x} = 16.1$$

+ Example – One Sample Z-Interval for a Population Mean

A bottling machine is operating with a standard deviation of 0.12 ounce. Suppose that in an SRS of 36 bottles the machine inserted an average of 16.1 ounces into each bottle.

B) Give an interval within which we are 95% certain that the mean lies.

For samples of size 36, the sample means are approximately normally distributed with a standard deviation of

$$\sigma_x = \frac{\sigma}{\sqrt{n}} = \frac{0.12}{\sqrt{36}} = 0.02$$

We want to use

$$\bar{x} \pm z^* \sigma_x$$

$$16.1 \pm 1.96(0.02)$$

$$= (16.0608, 16.1392)$$

Choosing the Sample Size

The margin of error *ME* of the confidence interval for the population mean μ is

$$z^* \cdot \frac{\sigma}{\sqrt{n}}$$

We determine a sample size for a desired margin of error when estimating a mean in much the same way we did when estimating a proportion.

Choosing Sample Size for a Desired Margin of Error When Estimating μ

To determine the sample size *n* that will yield a level *C* confidence interval for a population mean with a specified margin of error *ME*:

- Get a reasonable value for the population standard deviation σ from an earlier or pilot study.
- Find the critical value z^* from a standard Normal curve for confidence level *C*.
- Set the expression for the margin of error to be less than or equal to *ME* and solve for *n*:

$$n \geq \left(\frac{z^* \sigma}{ME}\right)^2$$

Example: How Many Monkeys?

Researchers would like to estimate the mean cholesterol level μ of a particular variety of monkey that is often used in laboratory experiments. They would like their estimate to be within 1 milligram per deciliter (mg/dl) of the true value of μ at a 95% confidence level. A previous study involving this variety of monkey suggests that the standard deviation of cholesterol level is about 5 mg/dl.

- ✓ The critical value for 95% confidence is $z^* = 1.96$.
- ✓ We will use $\sigma = 5$ as our best guess for the standard deviation.

$$n \geq \left(\frac{1.96\,(5)}{1}\right)^2$$

$$n > 96.04$$

We round up to 97 monkeys to ensure the margin of error is no more than 1 mg/dl at 95% confidence.

Checkpoint

1) At a certain plant, batteries are produced with a life expectancy that has a variance of 5.76 months squared. Suppose the mean life expectancy in an SRS of 64 batteries is 12.35 months.

Find a 90% confidence interval estimate of life expectancy for all batteries produced at this plant.

The standard deviation of the population is $\sigma = \sqrt{5.76} = 2.4$

So, the standard deviation of the sample mean is $\sigma_x = \dfrac{2.4}{\sqrt{64}} = 0.3$

$$\bar{x} \pm z^* \sigma_x$$

$$12.35 \pm 1.65(0.3)$$

$$= 12.35 \pm 0.4935$$

$$= (11.8565, 12.8435)$$

Checkpoint

2)To assess the accuracy of a laboratory scale, a standard weight known to weigh 10 grams is weighed repeatedly. The scale readings are Normally distributed with unknown mean (this mean is 10 grams if the scale has no bias). In previous studies, the standard deviation of the scale readings has been about 0.0002 gram. How many measurements must be averaged to get a margin of error of 0.0001 with 98% confidence? Show your work.

$$n \geq \left(\frac{2.33\ (0.0002)}{0.0001}\right)^2$$

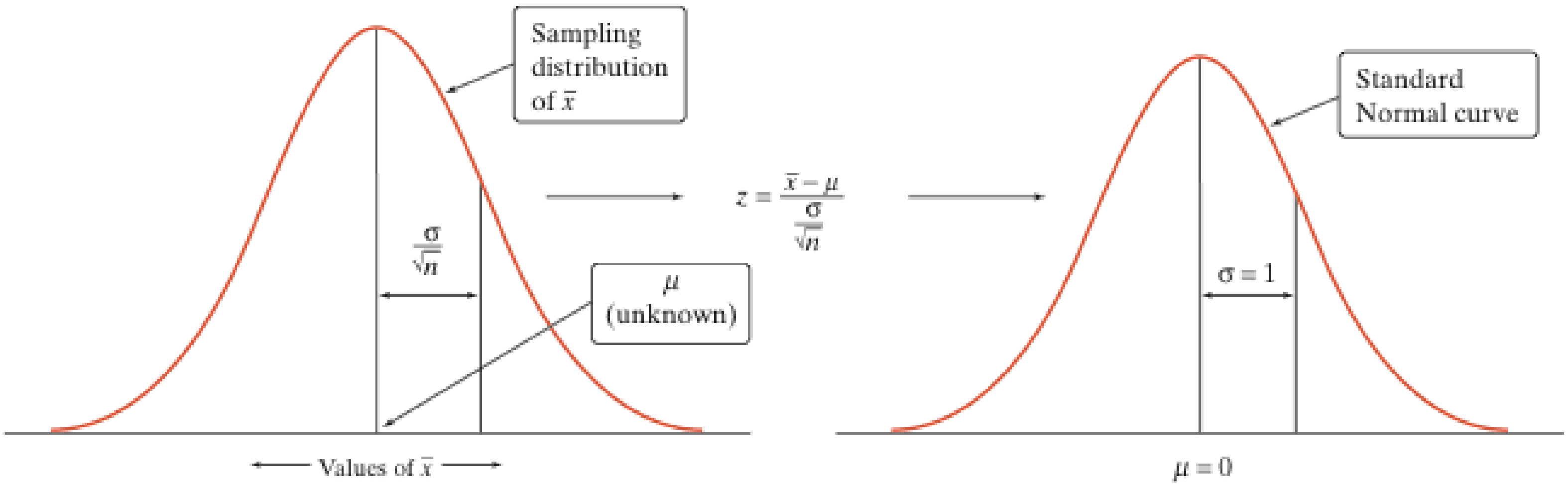
$$n \geq 21.7156$$

$$n \geq 22$$

When σ is Unknown: The *t* Distributions

When the sampling distribution of $\bar{x}$ is close to Normal, we can find probabilities involving $\bar{x}$ by standardizing:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$



When we don't know σ, we can estimate it using the sample standard deviation s_x. What happens when we standardize?

$$?? = \frac{\bar{x} - \mu}{s_x/\sqrt{n}}$$

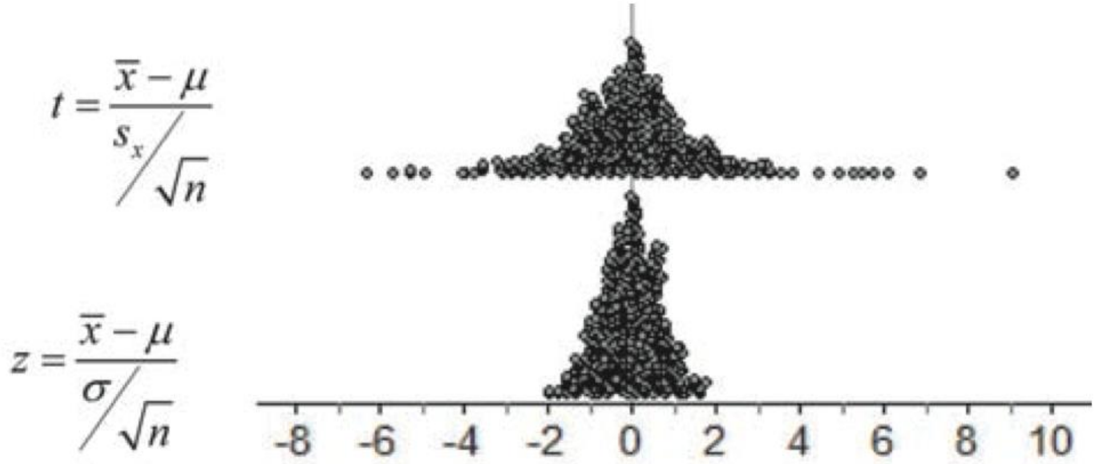
This new statistic does *not* have a Normal distribution!

When σ is Unknown: The *t* Distributions

When we standardize based on the sample standard deviation s_x, our statistic has a new distribution called a ***t* distribution**.

It has a *different shape* than the standard Normal curve:

- ✓ It is symmetric with a single peak at 0,
- ✓ However, it has much more area in the tails.



Like any standardized statistic, t tells us how far $\bar{x}$ is from its mean μ in standard deviation units.

However, there is a different *t* distribution for each sample size, specified by its **degrees of freedom (df)**.

The *t* Distributions; Degrees of Freedom

When we perform inference about a population mean μ using a *t* distribution, the appropriate degrees of freedom are found by subtracting 1 from the sample size *n*, making df = *n* - 1. We will write the *t* distribution with *n* - 1 degrees of freedom as t_{n-1}.

The *t* Distributions; Degrees of Freedom

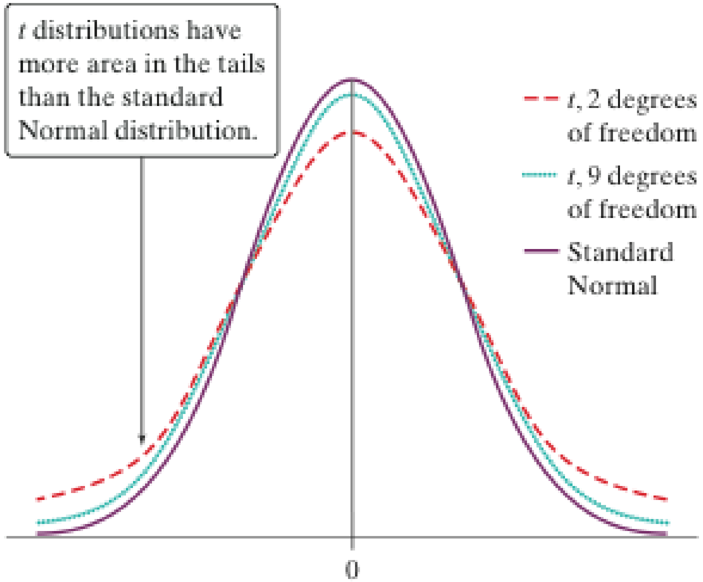
Draw an SRS of size *n* from a large population that has a Normal distribution with mean μ and standard deviation σ. The statistic

$$t = \frac{\bar{x} - \mu}{s_x / \sqrt{n}}$$

has the ***t* distribution** with **degrees of freedom** df = n – 1. The statistic will have approximately a t_{n-1} distribution as long as the sampling distribution is close to Normal.

The *t* Distributions; Degrees of Freedom

When comparing the density curves of the standard Normal distribution and *t* distributions, several facts are apparent:



✓The density curves of the *t* distributions are similar in shape to the standard Normal curve.

✓The spread of the *t* distributions is a bit greater than that of the standard Normal distribution.

✓The *t* distributions have more probability in the tails and less in the center than does the standard Normal.

✓As the degrees of freedom increase, the *t* density curve approaches the standard Normal curve ever more closely.

We can use invT on the calculator to determine critical values *t** for *t* distributions with different degrees of freedom.

Using the calculator to Find Critical t^* Values

Suppose you want to construct a 95% confidence interval for the mean μ of a Normal population based on an SRS of size n = 12. What critical t^* should you use?

In the calculator, we use the area as 0.975 and the degrees of freedom as 11.

The desired critical value is $t^* = 2.201$.

Using Graphing Calculator to Find Critical *t** Values

Suppose you want to construct a 95% confidence interval for the mean μ of a Normal population based on an SRS of size $n = 12$. What critical t^* should you use?

1) Go to 2nd VARS
2) Choose 4: invT (0.5 + C/2, n-1)
3) invT(0.975, 11)

The desired critical value is $t^* = 2.201$.

Checkpoint

- Find the critical value t^* that you would use for a confidence interval for a population mean μ in each of the following situations.

1. A 98% confidence interval based on n = 22 observations.

   t* = 2.518

2. A 90% confidence interval from an SRS of 10 observations.

   t* = 1.833

3. A 95% confidence interval from a sample of size 7.

   t* = 2.447

Constructing a Confidence Interval for μ

When the conditions for inference are satisfied, the sampling distribution for $\bar{x}$ has roughly a Normal distribution. Because we don't know σ, we estimate it by the sample standard deviation s_x.

Definition:

The **standard error of the sample mean** $\bar{x}$ is $\frac{s_x}{\sqrt{n}}$, where s_x is the sample standard deviation. It describes how far $\bar{x}$ will be from μ, on average, in repeated SRSs of size n.

To construct a confidence interval for μ,

✓Replace the standard deviation of $\bar{x}$ by its standard error in the formula for the one-sample z interval for a population mean.

✓Use critical values from the t distribution with n - 1 degrees of freedom in place of the z critical values. That is,

$$\text{statistic} \pm (\text{critical value}) \cdot (\text{standard deviation of statistic})$$

$$= \bar{x} \pm t^* \frac{s_x}{\sqrt{n}}$$

One-Sample t Interval for a Population Mean

The **one-sample *t* interval for a population mean** is similar in both reasoning and computational detail to the one-sample z interval for a population proportion. As before, we have to verify three important conditions before we estimate a population mean.

Conditions for Inference about a Population Mean

- **Random**: The data come from a random sample of size n from the population of interest or a randomized experiment.
- **Normal**: The population has a Normal distribution or the sample size is large ($n \geq 30$).
- **Independent**: The method for calculating a confidence interval assumes that individual observations are independent. To keep the calculations reasonably accurate when we sample without replacement from a finite population, we should check the *10% condition*: verify that the sample size is no more than 1/10 of the population size.

Example: Video Screen Tension

A manufacturer of high-resolution video terminals must control the tension on the mesh of fine wires that lies behind the surface of the viewing screen. Too much tension will tear the mesh, and too little will allow wrinkles. The tension is measured by an electrical device with output readings in millivolts (mV). Some variation is inherent in the production process. Here are the tension readings from a random sample of 20 screens from a single day's production:

269.5	297.0	269.6	283.3	304.8	280.4	233.5
257.4	317.5	327.4	264.7	307.7	310.0	343.3
328.1	342.6	338.8	340.1	374.6	336.1	

Construct and interpret a 90% confidence interval for the mean tension μ of all the screens produced on this day.

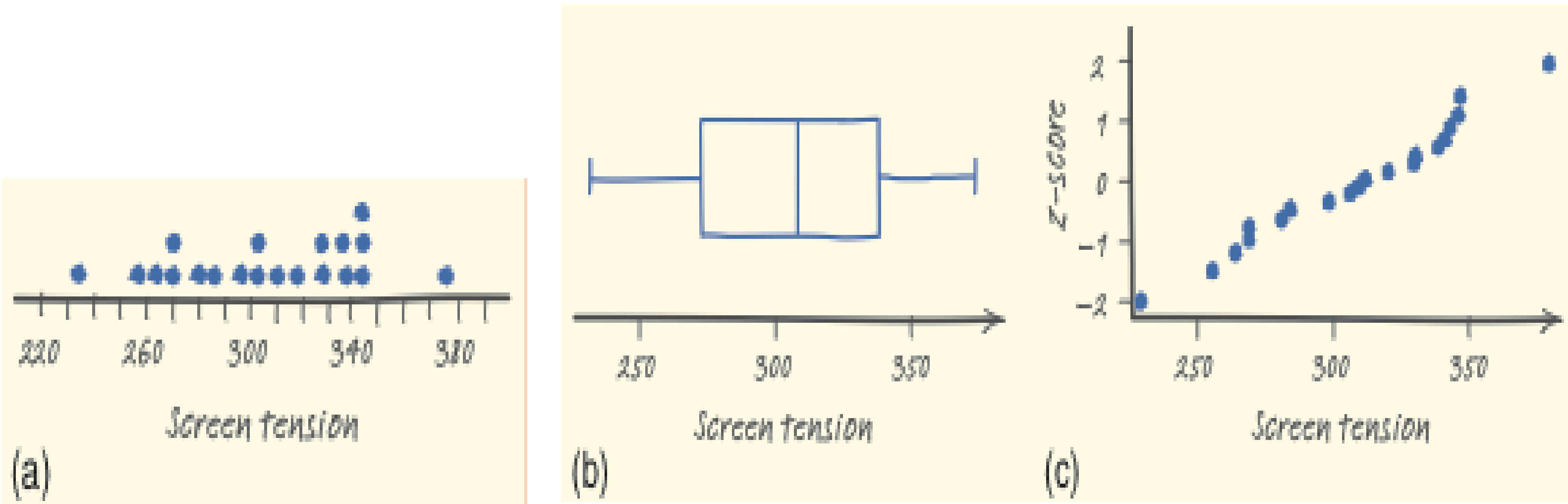
Example: Video Screen Tension

STATE: **We want to estimate the true mean tension μ of all the video terminals produced this day at a 90% confidence level.**

PLAN: If the conditions are met, we can use a one-sample t interval to estimate μ.

Random: We are told that the data come from a random sample of 20 screens from the population of all screens produced that day.

Normal: Since the sample size is small ($n < 30$), we must check whether it's reasonable to believe that the population distribution is Normal. Examine the distribution of the sample data.



These graphs give no reason to doubt the Normality of the population

Independent: Because we are sampling without replacement, we must check the 10% condition: we must assume that at least 10(20) = 200 video terminals were produced this day.

Example: Video Screen Tension

We want to estimate the true mean tension μ of all the video terminals produced this day at a 90% confidence level.

DO: Using our calculator, we find that the mean and standard deviation of the 20 screens in the sample are:

$$\bar{x} = 306.32 \text{ mV} \qquad \text{and} \qquad s_x = 36.21 \text{ mV}$$

Since n = 20, we use the t distribution with df = 19 to find the critical value.

From the calculator, we find t^* = 1.729.

Therefore, the 90% confidence interval for μ is:

$$\bar{x} \pm t^* \frac{s_x}{\sqrt{n}} = 306.32 \pm 1.729 \frac{36.21}{\sqrt{20}}$$

$$= 306.32 \pm 14$$

$$= (292.32,\ 320.32)$$

CONCLUDE: **We are 90% confident that the interval from 292.32 to 320.32 mV captures the true mean tension in the entire batch of video terminals produced that day.**

Example: Video Screen Tension

CONCLUDE: We are 90% confident that the interval from 292.32 to 320.32 mV captures the true mean tension in the entire batch of video terminals produced that day.

Example – Sample Size for a Mean

Ball bearings are manufactured by a process that results in a standard deviation in diameter of 0.025 inch. What sample size should be chosen if we wish to be 99% confident of knowing the diameter to within ±0.01 inch?

$$\sigma_x = \frac{\sigma}{\sqrt{n}} = \frac{0.025}{\sqrt{n}}$$

$$2.576\frac{0.025}{\sqrt{n}} \leq 0.01$$

$$\sqrt{n} \geq \frac{2.576(0.025)}{0.01}$$

$$n \geq 41.5, so\, n = 42$$

$$n = \left(\frac{z\sigma}{error}\right)^2$$

Checkpoint

- Biologists studying the healing of skin wounds measured the rate at which new cells closed a cut made in the skin of an anesthetized newt. Here are data from a random sample of 18 newts, measured in micrometers (millionths of a meter) per hour.

29 27 34 40 22 28 14 35 26 35 12 30 23 18
11 22 23 33

We want to estimate the mean healing rate μ with a 95% confidence interval.

1. Define the parameter of interest.

Population mean healing rate.

Checkpoint

2. What inference method will you use? Check that the conditions for using this procedure are met.

One-sample t interval for μ.

Random: The description says that the newts were randomly chosen.

Normal: We do not know if the data are Normal and there are fewer than 30 observations, so we graph the data to check its shape. Create a histogram on the calculator to check.

Independent: We have data on 18 newts. There are more than 180 newts, so this condition is met.

3. Construct a 95% confidence interval for μ. Show your method.

$$25.67 \pm 2.110 \frac{8.32}{\sqrt{18}}$$

$$= (21.53, 29.81)$$

4. Interpret your interval in context.

We are 95% confident that the interval from 21.53 to 29.81 micrometers per hour captures the true mean healing time for newts.

Using *t* Procedures Wisely

The stated confidence level of a one-sample *t* interval for μ is exactly correct when the population distribution is exactly Normal. No population of real data is exactly Normal. The usefulness of the *t* procedures in practice therefore depends on how strongly they are affected by lack of Normality.

Definition:

An inference procedure is called **robust** if the probability calculations involved in the procedure remain fairly accurate when a condition for using the procedures is violated.

Fortunately, the *t* procedures are quite robust against non-Normality of the population except when outliers or strong skewness are present. Larger samples improve the accuracy of critical values from the *t* distributions when the population is not Normal.

Using *t* Procedures Wisely

Except in the case of small samples, the condition that the data come from a random sample or randomized experiment is more important than the condition that the population distribution is Normal. Here are practical guidelines for the Normal condition when performing inference about a population mean.

Using One-Sample *t* Procedures: The Normal Condition

- *Sample size less than 15*: Use *t* procedures if the data appear close to Normal (roughly symmetric, single peak, no outliers). If the data are clearly skewed or if outliers are present, do not use *t*.
- *Sample size at least 15*: The *t* procedures can be used except in the presence of outliers or strong skewness.
- *Large samples*: The *t* procedures can be used even for clearly skewed distributions when the sample is large, roughly $n \geq 30$.

Checkpoint

1) You construct three 88% confidence intervals as follows:

A) A t-interval with 6 degrees of freedom.

B) A t-interval with 2 degrees of freedom

C) A z-interval

Assuming the mean and standard deviation are the same for all three intervals, (A, B, and C) arrange the intervals in order, from narrowest to widest.

C is more narrow than A, A is more narrow than B.

For t-intervals, critical values for any specific confidence level decrease as the degrees of freedom increases.

A z-interval is equivalent to a t-interval with “infinite” degrees of freedom.

+ Checkpoint

2) You are sampling from a population with a known standard deviation of 20 and want to construct a 95% confidence interval with a margin of error of no more than 4. What is the smallest sample that will produce such an interval?

We want a sample size n such that

$$(1.96)\left(\frac{20}{\sqrt{n}}\right) \leq 4$$

Solving as an inequality produces n = 96.04, so n should be 97.

Section 8.3
Estimating a Population Mean

Summary

In this section, we learned that…

- ✓ **Confidence intervals for the mean μ of a Normal population** are based on the sample mean of an SRS.
- ✓ If we somehow know σ, we use the *z* critical value and the standard Normal distribution to help calculate confidence intervals.
- ✓ The sample size needed to obtain a confidence interval with approximate margin of error ME for a population mean involves solving

$$z^* \frac{\sigma}{\sqrt{n}} \le ME$$

- ✓ In practice, we usually don't know σ. Replace the standard deviation of the sampling distribution with the **standard error** and use the *t* distribution with $n - 1$ **degrees of freedom (df)**.

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Section 8.3
Estimating a Population Mean

Summary

- ✓ There is a *t* distribution for every positive degrees of freedom. All are symmetric distributions similar in shape to the standard Normal distribution. The *t* distribution approaches the standard Normal distribution as the number of degrees of freedom increases.

- ✓ A level C confidence interval for the mean μ is given by the one-sample *t* interval

$$\bar{x} \pm t^* \frac{s_x}{\sqrt{n}}$$

- ✓ This inference procedure is approximately correct when these conditions are met: Random, Normal, Independent.

- ✓ The *t* procedures are relatively robust when the population is non-Normal, especially for larger sample sizes. The *t* procedures are not robust against outliers, however.